On the quantum capacity of the qubit depolarizing channel

based on arXiv:1701.03081 and a paper appearing May 14

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# Table of Contents

1. Qubit depolarizing channel
2. Quantum capacity
3. Approximate degradability and low-noise channels
4. Useful states and entanglement distillation
5. Conclusion
Assume that we have a qubit in a state $\rho$.

Fixing some basis $\{|0\rangle, |1\rangle\}$, we consider the following errors:

<table>
<thead>
<tr>
<th>error</th>
<th>action</th>
<th>operator</th>
</tr>
</thead>
<tbody>
<tr>
<td>bit flip</td>
<td>$</td>
<td>0\rangle \leftrightarrow</td>
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<td></td>
<td>$</td>
<td>1\rangle \leftrightarrow</td>
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<td>phase flip</td>
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<td>0\rangle \leftrightarrow</td>
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<td>$</td>
<td>1\rangle \leftrightarrow -</td>
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<tr>
<td>combined flip</td>
<td>$</td>
<td>0\rangle \leftrightarrow i</td>
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<td>$</td>
<td>1\rangle \leftrightarrow -i</td>
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Qubit depolarizing channel

- **Depolarizing channel** $\mathcal{D}_p$: each of the three errors (bit flip, phase flip, combined flip) occurs with the same probability $p/3$:

$$
\mathcal{D}_p(\rho) := (1 - p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z).
$$

- Alternatively: $\mathcal{D}_p$ replaces the input with the completely mixed state $\pi_2 := \frac{1}{2}I_2$ with ”probability” $q = 4p/3$:

$$
\mathcal{D}_p(\rho) = (1 - q)\rho + q\pi_2.
$$

- Exact expression for classical capacity of $\mathcal{D}_p$ is known. [King 2003]

- Exact expression for **quantum capacity of** $\mathcal{D}_p$ is unknown.
# Table of Contents

1. Qubit depolarizing channel

2. Quantum capacity

3. Approximate degradability and low-noise channels

4. Useful states and entanglement distillation

5. Conclusion
Entanglement generation

- Alice and Bob are connected by a quantum channel $\mathcal{N}: A \rightarrow B$ that they can use $n$ times.

- **Goal:** Generate entanglement in the form of $m_n$ ebits
  $$|\Phi\rangle \sim |00\rangle + |11\rangle.$$

- **Protocol:** Alice prepares a pure state in her lab, sends one half through $n$ copies of $\mathcal{N}$, Bob applies some decoding.

- **Quantum capacity** $Q(\mathcal{N})$: largest possible rate at which ebits can be generated with vanishing error.
Quantum capacity

▶ **LSD formula** [Lloyd 1997; Shor 2002; Devetak 2005]:

\[
Q(N) = \lim_{n \to \infty} \frac{1}{n} Q^{(1)}(N^\otimes n), \quad (\ast)
\]

where **channel coherent information** \(Q^{(1)}(N)\) is defined as

\[
Q^{(1)}(N) := \max_{|\psi\rangle_{A'}} I(A'|B)_{(\text{id} \otimes N)(\psi)},
\]

with the coherent information \(I(A|B)_{\rho} = S(B)_{\rho} - S(AB)_{\rho}\).

▶ **Hashing bound:** \(Q(N) \geq Q^{(1)}(N)\) [Devetak and Winter 2005]

▶ **Regularized formula** \((\ast)\) is in general **intractable to compute**, in particular for \(D_p\).

▶ Hence, we are interested in lower and upper bounds on \(Q(D_p)\).
Upper and lower bounds on $Q(D_p)$

Channel coherent information
Sutter et al. 2014; Smith and Smolin 2008
Upper and lower bounds on $Q(D_p)$
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- - - - Channel coherent information

Sutter et al. 2014; Smith and Smolin 2008

arXiv:1701.03081 (FL, N. Datta, G. Smith)

Approximate degradability bound

Useful states bound
Table of Contents

1. Qubit depolarizing channel

2. Quantum capacity

3. Approximate degradability and low-noise channels

4. Useful states and entanglement distillation

5. Conclusion
Low depolarizing noise

- For low depolarizing noise, i.e., $p$ close to 0, the depolarizing channel is "almost" the identity channel:

$$\mathcal{D}_p = (1 - p) \text{id} + \frac{p}{3} (\text{Pauli errors})$$

- Capacity of the identity channel is known: $Q(\text{id}) = \log d$.

- Particularly simple example of a degradable channel:

  degradable:
  \[
  \exists \mathcal{D} : B \rightarrow E \text{ s.t. } \quad \mathcal{N}^c = \mathcal{D} \circ \mathcal{N}
  \]

  $V_{\mathcal{N}} : A \rightarrow BE$ is an isometry s.t.
  \[
  \mathcal{N} = \text{Tr}_E (V \cdot V^\dagger), \\
  \mathcal{N}^c = \text{Tr}_B (V \cdot V^\dagger).
  \]
Degradability and quantum capacity

► Recall: Channel coherent information

\[ Q^{(1)}(\mathcal{N}) := \max \langle \psi \rangle_{A'} I(A' \rangle B)(\text{id} \otimes \mathcal{N})(\psi). \]

► Recall: LSD-formula

\[ Q(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} Q^{(1)}(\mathcal{N} \otimes n). \]

► Degradable channels: \( Q^{(1)}(\cdot) \) is weakly additive,

\[ Q^{(1)}(\mathcal{N} \otimes n) = nQ^{(1)}(\mathcal{N}) \quad \Longrightarrow \quad Q(\mathcal{N}) = Q^{(1)}(\mathcal{N}) \quad (*) \]

[Devetak and Shor 2005]

► Idea: Since \( D_p \) is degradable for \( p = 0 \), is \((*)\) approximately true for \( p \gtrsim 0 \)?
Continuity of quantum capacity

- What does it mean for two channels to be close?

- **Diamond norm:** For a linear map $\Phi$ with $d$-dimensional input,

  $$
  \| \Phi \|_\Diamond := \sup_X \frac{\| (\text{id}_d \otimes \Phi)(X) \|_1}{\| X \|_1}.
  $$

  - Operational meaning: $\frac{1}{2} \| \Phi - \Psi \|_\Diamond$ measures distinguishability of channels $\Phi$ and $\Psi$ (similar to trace distance and states).

- Quantum capacity is continuous with respect to the diamond norm [Leung and Smith 2009]:

  If $\| \mathcal{N} - \mathcal{M} \|_\Diamond \leq \epsilon$, then $|Q(\mathcal{N}) - Q(\mathcal{M})| \leq 8\epsilon \log |B| + 4h(\epsilon)$,

  where $h(\epsilon)$ is the binary entropy.
Low depolarizing noise and approximate degradability

1. Since $\|D_p - id\|_{\diamond} \leq p$, we expect $Q(D_p) \lesssim 1$ for small $p$.

2. However, continuity bound is not useful for quantitative bounds.

3. Recall: $\mathcal{N}$ is degradable if there is a $\mathcal{D}$ such that $\mathcal{N}^{c} = \mathcal{D} \circ \mathcal{N}$.

4. **Approximate degradability** [Sutter et al. 2014]  
   A channel $\mathcal{N}$ is $\eta$-degradable if there is another channel $\mathcal{D}$ s.t. 
   \[ \|\mathcal{N}^{c} - \mathcal{D} \circ \mathcal{N}\|_{\diamond} \leq \eta. \]

5. **Degradability parameter** $\text{dg}(\mathcal{N}) :=$ optimal such $\eta$

6. $\text{dg}(\mathcal{N})$ is the solution of an SDP (also yields optimal $\mathcal{D}$).
Approximate degradability and quantum capacity

- **Hashing bound:** \( Q(\mathcal{N}) \geq Q^{(1)}(\mathcal{N}) \) [Devetak and Winter 2005]

- This is optimal for **degradable channels:** [Devetak and Shor 2005]

\[
Q^{(1)}(\mathcal{N}^\otimes n) = nQ^{(1)}(\mathcal{N}) \implies Q(\mathcal{N}) = Q^{(1)}(\mathcal{N})
\]

- **\( \eta \)-degradable channels:**

\[
Q^{(1)}(\mathcal{N}) \leq Q(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + f(\eta, |E|),
\]

where \( f(\eta, |E|) = -\eta \log \eta + O(\eta) \).

- Recovers degradable case: \( f(\eta, |E|) \xrightarrow{\eta \to 0} 0 \).

- **Problem:** In the generic case, this is not a very useful approximation.
Approximate degradability and quantum capacity

**Problem:** In the generic case, this is not a very useful approximation.

\[
|Q(\mathcal{N}) - Q^{(1)}(\mathcal{N})| 
\approx -\eta \log \eta
\]

\[\eta\log \eta\]

infinite slope!
Approximate degradability and quantum capacity

**But:** If $\text{dg}(\mathcal{N}) = p^a$ for some underlying parameter $p$, and $a > 1$

$\Rightarrow$ approximation is tangent!

$$|Q(\mathcal{N}) - Q^{(1)}(\mathcal{N})| \\ \simeq -p^a \log p^a$$
Main results

- **Degradability:** $\text{dg}(\mathcal{D}_p) = O(p^2)$.

- **Quantum capacity:**
  $$Q(\mathcal{D}_p) = 1 - h(p) - p \log 3 + O(p^2 \log p^2).$$

- **Proof idea:** Guess degrading map $\mathcal{M}_p$ in $\|\mathcal{D}_p^c - \mathcal{M}_p \circ \mathcal{D}_p\|_\diamond$.

- **Intuition:** For small $p$, depolarizing channel is almost identity, so complementary channel is almost completely depolarizing.

- Let’s use the **complementary channel** itself as degrading map $\mathcal{M}_p$, and “give a little more to Bob”:
  $$\mathcal{M}_p = \mathcal{D}_{s(p)}^c \quad \text{with} \quad s(p) = p + ap^2.$$  

- For $a = 8/3$, this yields $\|\mathcal{D}_p^c - \mathcal{D}_{s(p)}^c \circ \mathcal{D}_p\|_\diamond = O(p^2)$. 
Usefulness of approximate degradability bound

- Approximate degradability might be useless in generic case.

- **However:** Extremely useful for channels with some underlying noise parameter such as $D_p$, for which $\text{dg}(D_p) = O(p^2)$.

- **Generalizations:**
  - Same quadratic ansatz works for any generalized Pauli channel $N$, proving $\text{dg}(N) = O(p^2)$.
  - For any ”low-noise channel” $N$ with $\| \text{id} - N \| \leq \epsilon$, we have $\|N^c - N^c \circ N\| \leq 2\epsilon^{1.5}$.

  $\rightarrow$ Low-noise channels are approximately degraded by their complementary channel!
Approximate degradability of depolarizing channel

\[ \text{where } s = p + \frac{8}{3}p^2 \]
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Qubit depolarizing channel</td>
</tr>
<tr>
<td>2</td>
<td>Quantum capacity</td>
</tr>
<tr>
<td>3</td>
<td>Approximate degradability and low-noise channels</td>
</tr>
<tr>
<td>4</td>
<td>Useful states and entanglement distillation</td>
</tr>
<tr>
<td>5</td>
<td>Conclusion</td>
</tr>
</tbody>
</table>
Upper and lower bounds on $Q(D_{p})$

Channel coherent information
- Sutter et al. 2014; Smith and Smolin 2008
- arXiv:1701.03081 (FL, N. Datta, G. Smith)

Approximate degradability bound

Useful states bound
Entanglement distillation: Convert "noisy" entanglement resource $\rho_{AB}$ shared between Alice and Bob into a "clean" form, e.g., ebits $|\Phi\rangle \sim |00\rangle + |11\rangle$.

Protocol: local operations (LO) and classical communication (CC) between Alice and Bob.

Recall that forward CC does not increase quantum capacity of a channel [Bennett et al. 1996; Barnum et al. 2000].

Restrict to forward CC from Alice to Bob only.

Distillable entanglement $D_\rightarrow(\rho_{AB})$: largest possible rate at which ebits can be distilled with vanishing error.
Entanglement distillation and entanglement generation

- **Entanglement distillation** and **entanglement generation** are just the **static** and **dynamic** side of the same coin! [Bennett et al. 1996]

- **egen → edist:**
  - Shared state $\rho_{AB}$ + forward LOCC $\rightarrow$ teleportation scheme [Bennett et al. 1993] $\rightarrow$ noisy channel $\mathcal{N}_\rho$.
  - egen code for $\mathcal{N}_\rho$ distills ebits from $\rho_{AB}$ $\rightarrow$ edist protocol.

- **edist → egen:**
  - ebit (in Alice’s lab) + channel $\mathcal{N}$ from Alice to Bob $\rightarrow$ shared Choi state $\tau_{\mathcal{N}}$.
  - edist protocol for $\tau_{\mathcal{N}}$ generates ebits through $\mathcal{N}$ $\rightarrow$ egen code.
Entanglement distillation and entanglement generation

- These two mappings (egen ↔ edist) are inverse to each other for so-called teleportation-simulable (TS) channels.

- **TS channel**: any output state can be obtained using a teleportation protocol run on the Choi state of the channel.

- Depolarizing channel is TS [Bennett et al. 1996]:

\[
Q(D_p) = D_\rightarrow(\tau_{D_p})
\]

- From now on, focus on the "static" resource \(\tau_{D_p}\) and \(D_\rightarrow(\tau_{D_p})\).
Useful states for entanglement distillation

► **Hashing bound** [Devetak and Winter 2005]

\[ D_\rightarrow(\rho_{AB}) \geq I(A\rangle B)_\rho. \]

► **But:** \( D_\rightarrow(\cdot) \) is also given by a regularized formula [Devetak and Winter 2005], and hence as intractable to compute as \( Q(\cdot) \).

► **Solution:** Identify ”useful” and ”useless” states which make the quantity \( D_\rightarrow(\cdot) \) behave nicely.

► **Useful states:** States \( \rho_{AB} \) for which hashing is optimal protocol:

\[ D_\rightarrow(\rho_{AB}) = I(A\rangle B)_\rho \geq 0. \]

► **Useless states:** Undistillable states \( \sigma_{AB} \), i.e., \( D_\rightarrow(\sigma_{AB}) = 0 \). They always have \( I(A\rangle B)_\sigma \leq 0 \).
Useful states for entanglement distillation

The **useful** and **useless** states for one-way entanglement distillation are given by **degradable** and **antidegradable** states.

\[
|\psi\rangle_{ABE} \text{ purifies } \rho_{AB}
\]

\[
\rho_{AB} \quad A \quad D
\]

\[
|\psi\rangle_{ABE}
\]

\[
\rho_{AE}
\]

\[
A
\]

\[
B
\]

\[
E
\]

degradable:

\[
\exists D : B \rightarrow E \text{ s.t. } \rho_{AE} = (\text{id}_A \otimes D)(\rho_{AB})
\]

antidegradable:

\[
\exists A : E \rightarrow B \text{ s.t. } \rho_{AB} = (\text{id}_A \otimes A)(\rho_{AE})
\]
Useful states for entanglement distillation

- **Key observation:** $D_\rightarrow(\cdot)$ is convex on degradable and antidegradable states!
  (proved by [Wolf and Pérez-García 2007] for quantum capacity)

- **Main result:** Decompose $\rho_{AB}$ into mixture of degradable ($\omega_i$) and antidegradable ($\sigma_i$) states,

$$\rho_{AB} = \sum_i p_i \omega_i + \sum_i p_i \sigma_i, \quad (*)$$

then we have the following upper bound:

$$D_\rightarrow(\rho_{AB}) \leq \sum_i p_i I(A\rangle B)_{\omega_i}. \quad (**)$$

- $(*)$ always exists, since every pure state is degradable.

- Minimizing $(**)$ over all pure-state decompositions of $\rho_{AB}$ yields entanglement of formation $E_F(\rho_{AB})$. 
Quantum capacity of depolarizing channel

► **Goal:** Apply this upper bound to Choi state $\tau_{D_p}$ of depolarizing channel to get upper bound on $Q(D_p) = D \rightarrow (\tau_{D_p})$.

► **Strategy:** Use symmetries of $\tau_{D_p}$!

► Choi state $\tau_{D_p}$ of depolarizing channel is an **isotropic state:**

1-parameter family of states $\{\text{Iso}(f) : f \in [0, 1]\}$ invariant under all local unitaries of the form $U \otimes U^*$. (here, $f = 1 - p$)

► The symmetry group $\{U \otimes U^* : U \text{ unitary}\}$ defines a bilateral twirl operation

$$\rho_{AB} \mapsto \int dU (U \otimes U^*) \rho_{AB} (U \otimes U^*)^\dagger = \text{Iso}(f_\rho)$$

where $f_\rho = \langle \Phi | \rho_{AB} | \Phi \rangle$. 

By the no-cloning theorem, $Q(D_p) = 0$ for $p \geq 1/4$.

For $p \leq 1/4$, the $U \otimes U^*$-symmetry of $\tau_{D_p}$ allows us to restrict to degradable states that "twirl" to $\tau_{D_p}$:

$$D \rightarrow (\tau_{D_p}) \leq \inf \{ I(A\rangle B)_{\rho} : \rho_{AB} \text{ degradable, } \langle \Phi | \rho_{AB} | \Phi \rangle = 1 - p \}.$$  

Bad news: This is a non-convex optimization problem, since the set of degradable states is not convex.

Good news: We can solve it numerically for low dimensions (qubits, qutrits) → yields advertised upper bound for $d = 2$!
## Table of Contents

1. Qubit depolarizing channel
2. Quantum capacity
3. Approximate degradability and low-noise channels
4. Useful states and entanglement distillation
5. Conclusion
Conclusion

- Quantum capacity of depolarizing channel is unknown.

- Best upper bound for low depolarizing noise: Approximate degradability bound.

- Approximates desirable properties of degradable channels.

- Best upper bound for high depolarizing noise: Useful states bound.

- Based on decomposition of Choi state into degradable and antidegradable parts.
References


Thank you very much for your attention!